ARKIV FÖR FYSIK Band 12 nr 28

Communicated 5 June 1957 by Oskar Klein and Bengt Edlén

Cross sections for bremsstrahlung and pair creation involving polarized electrons and photons

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With 23 figures in the text

SUMMARY

The differential cross sections for bremsstrahlung and pair creation involving arbitrarily polarized electrons and linearly and circularly polarized photons are computed. Numerical calculations are made and the cross sections are determined as functions of the angle between the scattering planes for circularly polarized photons, definite scattering angles and for four different energies and nine combinations of the polarization directions of the electrons as this is enough to determine the cross sections for arbitrarily polarized electrons. From the curves thus obtained we plot the maximum value of the cross sections as function of the energy of the incoming particle.

1. Introduction

The problem of producing and detecting polarized electrons has been dealt with by several authors. Recently a thorough review of the situation in theory and experiment has been given by Tolhoek [1], whose paper also contains a complete list of references. Up till now most of the formulae for phenomena containing photons have not contained polarization effects (cf., however, [2]), mainly because it has been difficult to produce polarized beams of electrons with high enough intensity. However, the work of Lee and Yang [3] on parity conservation indicates that the electrons emitted in β-decay will possibly be longitudinally polarized. Recent experiments [4, 5] seem to confirm this fact. It is thus possible to obtain beams of polarized electrons with an intensity high enough to perform experiments—for example, bremsstrahlung studies [6].

It is the aim of this paper to obtain the cross sections for bremsstrahlung and pair creation of polarized Dirac particles and of polarized photons. A paper on this subject, dealing with bremsstrahlung in the forward direction, has already appeared ([7]; cf. also [8]).

2. Field operators and S-matrix[1]

The incoming and outgoing electrons and photons are described by operators $\psi^{(0)}(x)$ and $A_\mu^{(0)}(x)$, which fulfill the equations

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[1] Cf. e.g. G. Källén, Quantenelektrodynamik, Handbuch der Physik 5 (to appear).

$$\left(\gamma \frac{\partial}{\partial x} + m\right) \psi^{(0)}(x) = 0 \tag{1}$$

and

$$\Box\, A_\mu^{(0)}(x) = 0. \tag{2}$$

The solution of (2) is written in the usual way

$$A_\mu^{(0)}(x) = \frac{1}{\sqrt{V}} \sum_{\bar{k}} \frac{1}{\sqrt{2\,\omega}} [e^{ikx} e_\mu a(\bar{k}) + e^{-ikx} e_\mu^* a^*(\bar{k})], \tag{3}$$

where $kx = \bar{k} \cdot \bar{x} - k_0 x_0$, $k_0 = \omega = |\bar{k}|$ and a, a^* are the well-known absorption and emission operators. The direction of polarization of the photon is given by the vector e and as the transversal photons represent the physical interesting solution of (2) we may choose e orthogonal to the momentum vector $\bar{k}$ of the photon and put $e_4 = 0$.

For the solution of (1) we have

$$\psi_\alpha^{(0)}(x) = \frac{1}{\sqrt{V}} \sum_{\bar{q}} [e^{i(\bar{q}\cdot\bar{x} - Ex_0)} u_\alpha^{(r)(+)}(\bar{q})\, a^{(r)}(\bar{q}) + e^{i(q\cdot\bar{x} + Ex_0)} u_\alpha^{(r)(-)}(\bar{q})\, b^{(r)}(\bar{q})]. \tag{4}$$

Here $a^{(r)}$ and $b^{(r)}$ are absorption and emission operators for the electrons and positrons respectively. In this case the polarization directions are indicated by the index r on the matrices $u^{(r)}(\bar{q})$. In the special representation where the γ-matrices are given by

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \tag{5}$$

the u-matrices have the following form:

	$u^{(+)}(\bar{q}) \cdot \sqrt{\frac{2E}{m+E}}$		$u^{(-)}(\bar{q}) \cdot \sqrt{\frac{2E}{m+E}}$	
r / α	1	2	1	2
1	1	0	$-\frac{q_z}{m+E}$	$\frac{-q_x + i q_y}{m+E}$
2	0	1	$-\frac{q_x + i q_y}{m+E}$	$\frac{q_z}{m+E}$
3	$\frac{q_z}{m+E}$	$\frac{q_x - i q_y}{m+E}$	1	0
4	$\frac{q_x + i q_y}{m+E}$	$-\frac{q_z}{m+E}$	0	1

(6)

$u^{(1)(+)}(\bar{q})$ and $u^{(2)(+)}(\bar{q})$ correspond to electrons with the spin preferably in the $\pm z$-direction respectively. However, we want the wave function to correspond to a spin direction making an arbitrary angle α with the z-axis. That may be obtained,

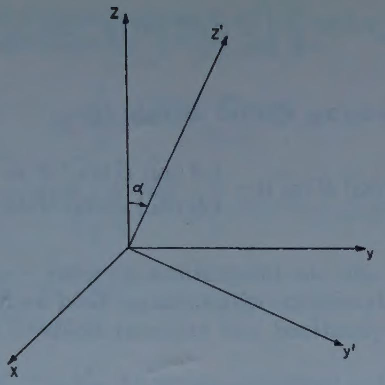

Fig. 1.

for example, by a rotation through the angle $-\alpha$ around the x-axis from the system x, y', z' (the spin points in the z'-direction) to the system x, y, z (Fig. 1). We then get for the new matrix $u^{(r)\prime}$

$$u^{(r)\prime}(\bar{q}) = T\, u^{(r)}(\bar{q}), \tag{7}$$

where T is the unitary matrix, which may, cf. [9], simply be written

$$T = e^{\frac{1}{2}\alpha\gamma_2\gamma_3}. \tag{8}$$

Here γ_2, γ_3 are the γ-matrices given by (5). The result of the operation with T is

$$\sqrt{\frac{2E}{m+E}}\, u^{(r)\prime}(\bar{q}) = \left\{ \begin{matrix} \cos\frac{\alpha}{2}, \\ i\sin\frac{\alpha}{2}, \\ \frac{1}{m+E}\left\{ q_y \sin\frac{\alpha}{2} + q_z \cos\frac{\alpha}{2} + i\, q_x \sin\frac{\alpha}{2} \right\}, \\ \frac{1}{m+E}\left\{ q_x \cos\frac{\alpha}{2} + i\left[q_y \cos\frac{\alpha}{2} - q_z \sin\frac{\alpha}{2} \right] \right\}. \end{matrix} \right\} \tag{9}$$

For $\alpha = 0$, that gives $u^{(1)}(\bar{q})$ and for $\alpha = \pi$, $i u^{(2)}(\bar{q})$.

The cross sections are obtained from the S-matrix, which is written as an expansion in the coupling constant

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dx_0' \ldots dx_0^{(n)}\, P\left(H^{(1)}(x_0') \ldots H^{(1)}(x_0^{(n)})\right), \tag{10}$$

where $$H^{(1)}(x_0) = -\frac{i e}{2}\int d^3 x\, A^{(0)}_\mu(x)\,[\bar{\psi}^{(0)}(x),\, \gamma_\mu \psi^{(0)}(x)] \tag{11}$$

and P means the time ordering symbol defined by

$$P(A(x_0')\,B(x_0'')) = \begin{cases} A(x_0')\,B(x_0'') \text{ if } x_0' > x_0'', \\ B(x_0'')\,A(x_0') \text{ if } x_0'' > x_0'. \end{cases} \tag{12}$$

In (11) $d^3 x = dx_1 dx_2 dx_3$ and the integrations go from $-\infty$ to $+\infty$ in all variables. As we are interested in interactions with external field we have to regard the operator $A^{(0)}_\mu(x)$ as a sum of the quantized and external fields

$$A^{(0)}_\mu(x) = A^{(0)}_\mu(x) + A^e_\mu(x). \tag{13}$$

This means that we expand not only in the radiation field but also in powers of the external field. For heavy nuclei (high values of Z) and (or) low energies, this expansion might not be appropriate.

We now take the matrix element of S between the states $|n\rangle$ and $|n'\rangle$ and write

$$\langle n'|S|n\rangle = \delta_{nn'} + \langle n'|R|n\rangle\,\delta(p_0' - p_0). \tag{14}$$

The transition probability per unit time between the states is then, as is well known, given by

$$W = \frac{1}{2\pi}|\langle n'|R|n\rangle|^2\,\delta(p_0' - p_0) \tag{15}$$

and from that we obtain the cross section by summing over the energies of the the final state and dividing by the intensity of incoming particles.

3. Cross section for bremsstrahlung

We now consider the special case where the state $|n\rangle$ in (14) corresponds to one incoming electron with momentum q and $|n'\rangle$ to one outgoing electron and one photon with momenta q' and k respectively. The first non-vanishing term of the S-matrix (10) then gives, according to well known rules of calculation,

$$\langle q'\,k|S|q\rangle = -\frac{e^2}{2}\int dx'\,dx''\,\langle q'|\bar{\psi}^{(0)}(x'')|0\rangle\,\gamma_\nu\,S_F(x''-x')\,\gamma_\mu \times$$

$$\times\langle 0|\psi^{(0)}(x')|q\rangle\,[\langle k|A^{(0)}_\nu(x'')|0\rangle\,A^e_\mu(x') + \langle k|A^{(0)}_\mu(x')|0\rangle\,A^e_\nu(x'')], \tag{16}$$

where $$S_F(x) = -\frac{2i}{(2\pi)^4}\int dp\, e^{ipx}\,(i\gamma p - m)\left[P\frac{1}{p^2+m^2} + i\pi\,\delta(p^2+m^2)\right]. \tag{17}$$

We now insert the Fourier representation of $A_\mu^{(0)}(x)$ and $\psi^{(0)}(x)$ given by (3) and (4) and the corresponding expression for $\bar{\psi}^{(0)}(x)$ and get

$$\begin{aligned}\langle q'\,k\,|\,S\,|\,q\rangle &= \frac{i\,2\pi\,e^2}{V^{3/2}\sqrt{2\,\omega}}\,\bar{u}^{(r')(+)}(\bar{q}')\Bigl[\gamma_\nu\,\frac{i\,\gamma\,(q-k)-m}{-2\,q\,k}\,\gamma\,e^* + \\ &\quad + \gamma\,e^*\,\frac{i\,\gamma\,(q'+k)-m}{2\,q'\,k}\,\gamma_\nu\Bigr]\,u^{(r)(+)}(\bar{q})\,A_\nu^e\,(\bar{q}-\bar{q}'-\bar{k})\,\delta\,(q_0-q_0'-k_0) \\ &= \frac{i\,2\pi\,e^2\,\delta\,(q_0-q_0'-k_0)}{V^{3/2}\sqrt{2\,\omega}}\langle q'\,k\,|\,M_\nu\,|\,q\rangle\,A_\nu^e\,(\bar{q}-\bar{q}'-\bar{k}),\end{aligned} \tag{18}$$

where $q\,k=\bar{q}\cdot\bar{k}-q_0 k_0$, $\gamma\,e^*=\gamma_\lambda e_\lambda^*$. As we are interested in the case when the external field is a Coulomb field with charge Ze, we put

$$A_\nu^e\,(\bar{Q}) = i\,\delta_{\nu 4}\,\frac{Z\,e}{\bar{Q}^2}. \tag{19}$$

We now have to calculate the matrix element $\langle q'k\,|\,M_4\,|\,q\rangle$ and to do that we insert for $u^{(r)(+)}(\bar{q})$ the explicit expression for $u^{(1)(+)}(\bar{q})$ given by (6) and for $u^{(r')(+)}(\bar{q}')$ the expression (9). That means that we assume the incoming electron to be polarized in the z-direction and the outgoing electron to be polarized in the z'-direction, where the angle between z and z' is equal to α. Obviously this is no limitation on the generality of our result.

Straightforward calculations now give

$$\langle q'\,k\,|\,M_4\,|\,q\rangle = \frac{i}{2}\left[(A_x+i\,A_y)\sin\frac{\alpha}{2}+(C+i\,A_z)\cos\frac{\alpha}{2}\right]\sqrt{\frac{(m+E)\,(m+E')}{4\,E\,E'}}, \tag{20}$$

where

$$\begin{aligned}\bar{A} &= \frac{m}{q\,k}\Bigl\{-\bar{e}^*\times\bar{k}+\frac{\omega}{m+E}\,\bar{e}^*\times\bar{q}-\frac{\omega}{m+E'}\,\bar{e}^*\times\bar{q}' + \\ &\quad + \frac{(\bar{e}^*\cdot\bar{q})\,(\bar{q}'\times\bar{k}-2\,\bar{q}'\times\bar{q})+(\bar{q}\cdot\bar{k})\,\bar{e}^*\times\bar{q}'+\bar{q}'\,(\bar{e}^*\cdot[\bar{q}\times\bar{k}])}{(m+E)\,(m+E')}\Bigr\} + \\ &\quad + \frac{m}{q'\,k}\Bigl\{\bar{e}^*\times\bar{k}+\frac{\omega}{m+E}\,\bar{e}^*\times\bar{q}-\frac{\omega}{m+E'}\,\bar{e}^*\times\bar{q}' - \\ &\quad - \frac{(\bar{e}^*\cdot\bar{q})\,\bar{q}'\times\bar{k}-2\,(\bar{e}^*\cdot\bar{q}')\,\bar{q}'\times\bar{q}+(\bar{q}\cdot\bar{k})\,\bar{e}^*\times\bar{q}'+\bar{q}'\,(\bar{e}^*\cdot[\bar{q}\times\bar{k}])}{(m+E)\,(m+E')}\Bigr\},\end{aligned} \tag{21}$$

$$\begin{aligned}C &= \frac{m}{q\,k}\Bigl\{(\bar{e}^*\cdot\bar{q})\left[-2+\frac{\omega}{m+E}+\frac{\bar{q}'\cdot\bar{k}-2\,\bar{q}\cdot\bar{q}'}{(m+E)\,(m+E')}\right] + \\ &\quad + (\bar{e}^*\cdot\bar{q}')\left[\frac{\omega}{m+E'}-\frac{\bar{q}\cdot\bar{k}}{(m+E)\,(m+E')}\right]\Bigr\} +\end{aligned}$$

$$+\frac{m}{q'k}\left\{(\bar{e}^*\cdot\bar{q})\left[\frac{\omega}{m+E}-\frac{\bar{q}'\cdot\bar{k}}{(m+E)\,(m+E')}\right]+\right.$$

$$\left.+(\bar{e}^*\cdot\bar{q}')\left[2+\frac{\omega}{m+E'}+\frac{\bar{q}\cdot\bar{k}+2\,\bar{q}\cdot\bar{q}'}{(m+E)\,(m+E')}\right]\right\}. \tag{22}$$

To obtain the scattering cross section we calculate $|\langle q'k|M_4|q\rangle|^2$ and write

$$|\langle q'\,k\,|\,M_4\,|\,q\,\rangle|^2=\frac{(m+E)\,(m+E')}{16\,E\,E'}\cdot R. \tag{23}$$

As the intensity of incoming particles is

$$I=\frac{|\bar{q}|}{E\,V} \tag{24}$$

and the density of final states is given by

$$\varrho_F=\varrho(\bar{q}')\,\varrho(\bar{k})\,d\,\omega\,d\,E', \tag{25}$$

where $$\varrho\,(\bar{q}')=\frac{V}{(2\,\pi)^3}\,|\,\bar{q}'\,|\,E'\,d\,\Omega_{q'},\qquad \varrho\,(\bar{k})=\frac{V}{(2\,\pi)^3}\,\omega^2\,d\,\Omega_k, \tag{26}$$

we obtain for the cross section for bremsstrahlung after integration over energies E' of the outgoing electron ($E'=E-\omega$ everywhere)

$$d\,\sigma=\frac{Z^2\,e^6}{(4\,\pi)^5\,m^2}\cdot\frac{|\,\bar{q}'\,|\,\omega\,(m+E)\,(m+E')}{|\,\bar{q}\,|\,[(\bar{q}-\bar{q}'-\bar{k})^2]^2}\cdot R\cdot d\,\omega\,d\,\Omega_k\,d\,\Omega_{q'}$$

$$=Z^2\,\phi\cdot\frac{d\,\omega}{\omega}\cdot d\,\Omega_k\,d\,\Omega_{q'}. \tag{27}$$

Here ϕ depends on eight parameters: the energies of the incoming electron and of the photon, the polarisation directions of both electrons and of the photon, and the directions of the momenta of the outgoing electron and photon. The latter will be described by the angle θ between $\bar{q}$ and $\bar{k}$, ϑ between $\bar{q}$ and $\bar{q}'$ and φ between the planes defined by $\bar{q},\bar{k}$ and $\bar{q},\bar{q}'$ (Fig. 2).

The polarization-dependent part of (27) is the function R, which is given by (23). To obtain explicit expressions for this function we first assume that the photon is linearly polarized. Then A_x, A_y, and A_z (20) are real and we get

$$R=\tfrac{1}{2}\,(A^2+C^2)+[-\tfrac{1}{2}\,(A^2+C^2)+A_z^2+C^2]\,\cos\alpha+(A_y\,A_z+A_x\,C)\,\sin\alpha. \tag{28}$$

There are two independent directions of the polarization vector $\bar{e}$ of the photon and we choose

I. $\bar{e}$ orthogonal to the $\bar{q},\bar{k}$-plane,

II. $\bar{e}$ in the $\bar{q},\bar{k}$-plane.

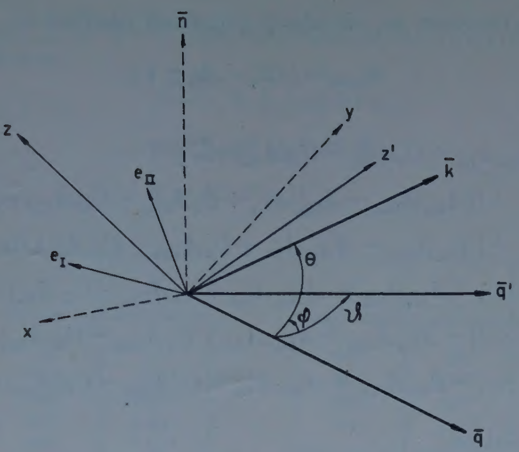

Fig. 2.

The corresponding functions we call ϕ_{I}, ϕ_{II}, R_{I}, R_{II}, $A_{\mathrm{I}x}$, $A_{\mathrm{II}x}$ etc.

The matrix element for circularly polarized photons is obtained from (20) by substituting for $\bar{e}$ a linear combination, $(1/\sqrt{2})(\bar{e}_{\mathrm{I}} \pm i\bar{e}_{\mathrm{II}})$, of the polarization vectors corresponding to the two cases above. Here the + signs refers to left circularly polarized photons. We get

$$\begin{aligned} R_{\mathrm{I}\pm i\mathrm{II}} = \tfrac{1}{2}[R_{\mathrm{I}} + R_{\mathrm{II}} &\pm (A_{\mathrm{I}x}A_{\mathrm{II}y} - A_{\mathrm{II}x}A_{\mathrm{I}y} + C_{\mathrm{I}}A_{\mathrm{II}z} - C_{\mathrm{II}}A_{\mathrm{I}z}) \pm \\ &\pm (-A_{\mathrm{I}x}A_{\mathrm{II}y} + A_{\mathrm{II}x}A_{\mathrm{I}y} + C_{\mathrm{I}}A_{\mathrm{II}z} - C_{\mathrm{II}}A_{\mathrm{I}z})\cos\alpha \pm \\ &\pm (A_{\mathrm{I}x}A_{\mathrm{II}z} - A_{\mathrm{II}x}A_{\mathrm{I}z} + C_{\mathrm{I}}A_{\mathrm{II}y} - C_{\mathrm{II}}A_{\mathrm{I}y})\sin\alpha]. \end{aligned} \tag{29}$$

According to Fig. 1 the coordinate system is fixed by the polarization directions of the electrons. This means that the components of $\bar{A}$ in (28) and (29) should be computed in nine coordinate systems corresponding to the nine different possibilities of choosing independent polarization directions. Instead of doing that we describe the spin direction, z, of the incoming electron by the angles β and γ, where γ is the angle between z and the normal $\bar{n} = \bar{q} \times \bar{q}'$ to the $\bar{q},\bar{q}'$-plane (Fig. 2) and β is the angle between the $\bar{n},\bar{q}$-plane and the $\bar{n},\bar{z}$-plane. The corresponding angles for z' are called β' and γ'. (Fig. 2 also shows how the coordinate system has to be choosen in this general case.) We now transform by successive rotations to a system x_0, y_0, z_0, with z_0 in the direction of $\bar{n}$ and x_0 in the direction of $\bar{q}$. The result is (writing index i instead of I or II)

$$\begin{aligned} R_i = \tfrac{1}{2}(A_i^2 + C_i^2) + \{&(-\tfrac{1}{2}(A_i^2 - C_i^2) + A_{iz_0}^2)\cos\gamma' + \\ &+ [(A_{iz_0}A_{ix_0} - C_i A_{iy_0})\cos\beta' + (A_{iz_0}A_{iy_0} + C_i A_{ix_0})\sin\beta']\sin\gamma'\}\cos\gamma + \\ &+ \{([(-\tfrac{1}{2}(A_i^2 - C_i^2) + A_{ix_0}^2)\cos\beta' + (A_{ix_0}A_{iy_0} - C_i A_{iz_0})\sin\beta']\cos\beta + \\ &+ [(-\tfrac{1}{2}(A_i^2 - C_i^2) + A_{iy_0}^2)\sin\beta' + (A_{ix_0}A_{iy_0} + C_i A_{iz_0})\cos\beta']\sin\beta)\sin\gamma' + \\ &+ [(A_{iz_0}A_{ix_0} + C_i A_{iy_0})\cos\beta + (A_{iz_0}A_{iy_0} - C_i A_{ix_0})\sin\beta]\cos\gamma'\}\sin\gamma. \end{aligned} \tag{30}$$

The corresponding function for circularly polarized photons is

$$R_{\mathrm{I}\pm i\mathrm{II}}=\tfrac{1}{2}\,(R_{\mathrm{I}}+R_{\mathrm{II}}\pm V), \tag{31}$$

where now

$$\begin{aligned}
V=(A_{\mathrm{I}x_0}A_{\mathrm{II}y_0}-A_{\mathrm{II}x_0}A_{\mathrm{I}y_0}+C_{\mathrm{I}}A_{\mathrm{II}z_0}-C_{\mathrm{II}}A_{\mathrm{I}z_0})\cos\gamma\,+&\\
+\,[(A_{\mathrm{I}y_0}A_{\mathrm{II}z_0}-A_{\mathrm{II}y_0}A_{\mathrm{I}z_0}+C_{\mathrm{I}}A_{\mathrm{II}x_0}-C_{\mathrm{II}}A_{\mathrm{I}x_0})\cos\beta\,+&\\
+\,(A_{\mathrm{I}z_0}A_{\mathrm{II}x_0}-A_{\mathrm{II}z_0}A_{\mathrm{I}x_0}+C_{\mathrm{I}}A_{\mathrm{II}y_0}-C_{\mathrm{II}}A_{\mathrm{I}y_0})\sin\beta]\sin\gamma\,+&\\
+\,(-A_{\mathrm{I}x_0}A_{\mathrm{II}y_0}+A_{\mathrm{II}x_0}A_{\mathrm{I}y_0}+C_{\mathrm{I}}A_{\mathrm{II}z_0}-C_{\mathrm{II}}A_{\mathrm{I}z_0})\cos\gamma'\,+&\\
+\,[(-A_{\mathrm{I}y_0}A_{\mathrm{II}z_0}+A_{\mathrm{II}y_0}A_{\mathrm{I}z_0}+C_{\mathrm{I}}A_{\mathrm{II}x_0}-C_{\mathrm{II}}A_{\mathrm{I}x_0})\cos\beta'\,+&\\
+\,(-A_{\mathrm{I}z_0}A_{\mathrm{II}x_0}+A_{\mathrm{II}z_0}A_{\mathrm{I}x_0}+C_{\mathrm{I}}A_{\mathrm{II}y_0}-C_{\mathrm{II}}A_{\mathrm{I}y_0})\sin\beta']\sin\gamma'.&
\end{aligned} \tag{32}$$

Introducing the notation

$$\left.\begin{aligned}
&a_0=\omega m\left(\frac{1}{q k}-\frac{1}{q' k}\right),\quad b_0=\omega m\left(\frac{1}{q k}+\frac{1}{q' k}\right),\\
&a_1=2\frac{m\,|\bar q|}{q k},\quad a_2=2\frac{m\,|\bar q'|}{q' k},\quad a_3=\frac{b_0\,|\bar q|}{m+E},\quad a_4=\frac{b_0\,|\bar q'|}{m+E'},\\
&a_5=\frac{a_0\,|\bar q|\,|\bar q'|}{(m+E)\,(m+E')},\quad a_6=\frac{m\,q^2\,|\bar q'|}{q k\,(m+E)\,(m+E')},\quad a_7=\frac{m\,q'^2\,|\bar q|}{q' k\,(m+E)\,(m+E')};\\
&\cos\theta''\;=-\sin\theta\cos\vartheta+\cos\theta\sin\vartheta\cos\varphi,\\
&\cos\theta'''=\quad\sin\theta\sin\vartheta+\cos\theta\cos\vartheta\cos\varphi,
\end{aligned}\right\} \tag{33}$$

the components of $\bar A$ and the function C are given by

$$\left.\begin{aligned}
A_{\mathrm{I}x_0}&=a_0\sin\theta+a_4\cos\varphi\sin\vartheta-a_5\cos\theta'',\\
A_{\mathrm{I}y_0}&=(-a_0\cos\theta+a_3-a_4\cos\vartheta)\cos\varphi+a_5\cos\theta''',\\
A_{\mathrm{I}z_0}&=\sin\varphi\,(-a_0\cos\theta+a_3-a_4\cos\vartheta+a_5\cos\theta\cos\vartheta+2a_7\sin^2\vartheta),\\
C_{\mathrm{I}}\;&=-\sin\vartheta\sin\varphi\,(a_4-a_5\cos\theta+a_2+2a_7\cos\vartheta)
\end{aligned}\right\} \tag{34}$$

and

$$\left.\begin{aligned}
A_{\mathrm{II}x_0}&=\sin\varphi\sin\vartheta\,(a_4\cos\theta-a_5),\\
A_{\mathrm{II}y_0}&=\sin\varphi\,(-a_0+a_3\cos\theta-a_4\cos\theta\cos\vartheta+a_5\cos\vartheta),\\
A_{\mathrm{II}z_0}&=a_0\cos\varphi-a_3\cos\theta\cos\varphi+a_4\cos\theta'''-a_5\cos\vartheta\cos\varphi\,-\\
&\qquad-2a_6\sin\theta\sin\vartheta-2a_7\sin\vartheta\cos\theta'',\\
C_{\mathrm{II}}\;&=a_1\sin\theta+a_2\cos\theta''-a_3\sin\theta+a_4\cos\theta''-a_5\sin\vartheta\cos\varphi\,+\\
&\qquad+2a_6\sin\theta\cos\vartheta+2a_7\cos\theta''\cos\vartheta.
\end{aligned}\right\} \tag{35}$$

The cross sections $\phi_{\mathrm{I} \pm i \mathrm{II}}$ have now been calculated as functions of φ for the following values of the parameters.

$$E = 1, 2, 4, \text{ and } 10\ m_e,$$
$$\omega = 0.5\,E \text{ and } 0.9\,E,$$
$$\theta = 5^\circ, \vartheta = 10^\circ.$$

γ	β	γ'	β'
0°		0°	
0°		90°	10°
0°		90°	100°
90°	0°	90°	100°
90°	90°	90°	10°

γ	β	γ'	β'
90°	90°	90°	100°
90°	180°	90°	10°
90°	180°	90°	190°
90°	180°	0°	

The choice of the values of θ and ϑ is wholly arbitrary. Calculations have also been made for other angles, but there seems to be only a difference in magnitude of the cross sections and the general behaviour is the same.

Figs. 3 and 4 show the curves obtained for backward polarization of both electrons ($\omega = 0.5\,E$, $\gamma = 90^\circ$, $\beta = 180^\circ$, $\gamma' = 90^\circ$, $\beta' = 190^\circ$). It follows from formulae (30) to (35) that all curves corresponding to $\gamma = \gamma' = 90^\circ$ are symmetric around $\varphi = 180^\circ$.

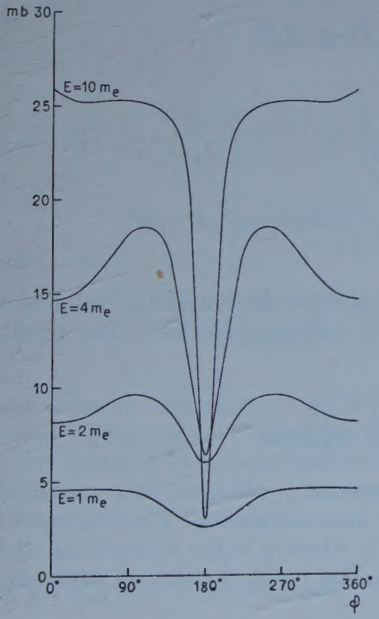

Fig. 3. $\phi_{\mathrm{I}-i\mathrm{II}}$, $\omega = 0.5\,E$, $\gamma = 90^\circ$, $\beta = 180^\circ$, $\gamma' = 90^\circ$, $\beta' = 190^\circ$.

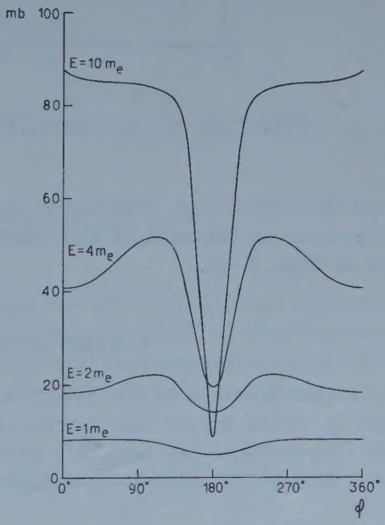

Fig. 4. $\phi_{\mathrm{I}+i\mathrm{II}}$, $\omega = 0.5\,E$, $\gamma = 90^\circ$, $\beta = 180^\circ$, $\gamma' = 90^\circ$, $\beta' = 190^\circ$.

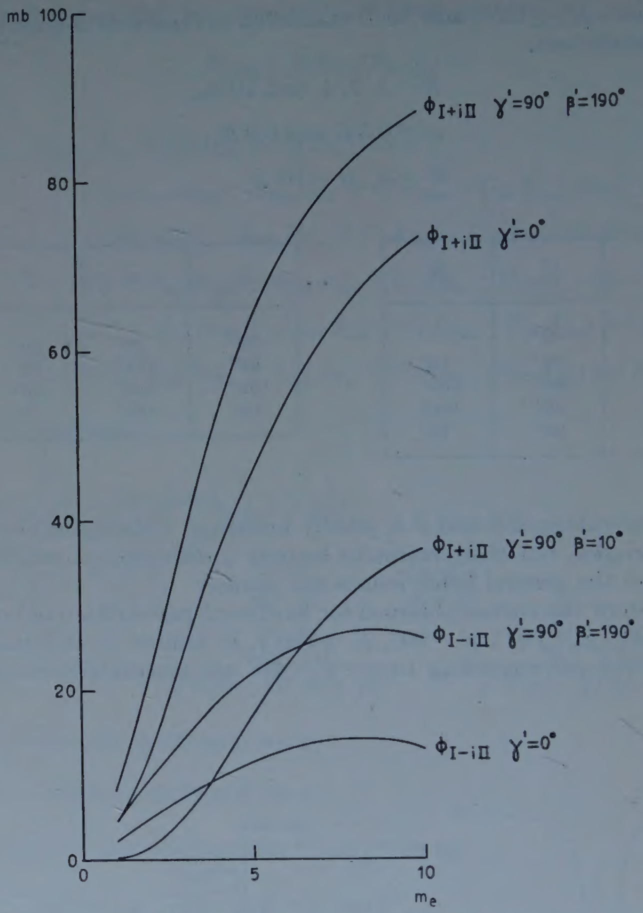

Fig. 5. $\omega = 0.5\,E$, $\gamma = 90°$, $\beta = 180°$; $\phi_{I-iII} < 0.02$, $\gamma' = 90°$, $\beta' = 10°$.

From the curves thus obtained we have finally plotted the maximum value of the cross sections as function of the energy E of the incoming electron. The result is shown in Figs. 5–12.

4. Cross section for pair creation

The matrix element of the S-matrix for pair creation is obtained in a way similar to that in which we got the matrix element for bremsstrahlung. We put the state $|n\rangle$ in (14) equal to $|k\rangle$ and $|n'\rangle$ equal to $|q, q'\rangle$, where q is the momentum of the electron and q' the momentum of the positron. The result written in momentum space is

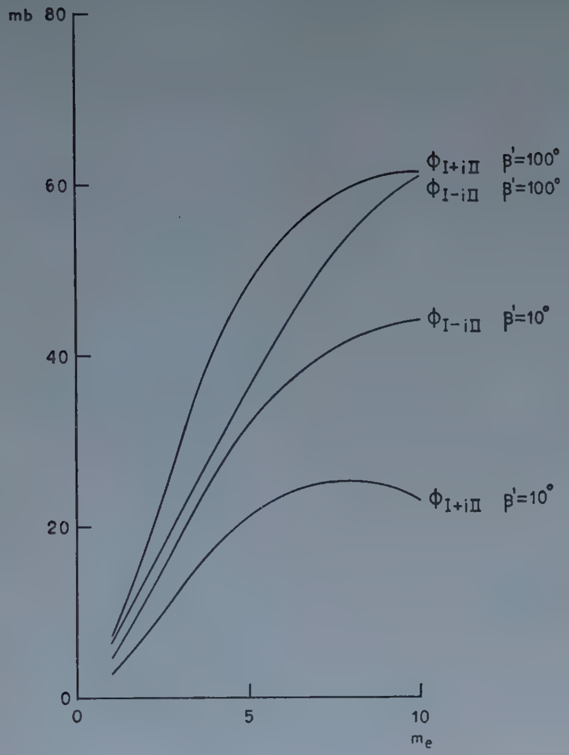

Fig. 6. $\omega = 0.5\,E$, $\gamma = 90°$, $\beta = 90°$, $\gamma' = 90°$.

$$\langle q, q' \,|\, S \,|\, k \rangle = \frac{i\,2\,\pi\,e^2}{V^{3/2}\sqrt{2\,\omega}}\,\bar{u}^{(r)(+)}(\bar{q})\left[\gamma_\mu \frac{i\,\gamma\,(q'-k)+m}{2\,q'\,k}\gamma\,e + \right.$$

$$\left. + \gamma\,e\,\frac{i\,\gamma\,(q-k)-m}{-2\,q\,k}\gamma_\mu\right] u^{(r')(-)}(-\bar{q}')\,A^e_\mu(\bar{k}-\bar{q}-\bar{q}')\,\delta(k_0-q_0-q_0')$$

$$= \frac{i\,2\,\pi\,e^2\,\delta(k_0-q_0-q_0')}{V^{3/2}\sqrt{2\,\omega}}\langle q, q' \,|\, M_\mu \,|\, k \rangle\, A^e_\mu(\bar{k}-\bar{q}-\bar{q}'). \tag{36}$$

In order to calculate $\langle q, q' | M_4 | k \rangle$ we insert for $u^{(r')(-)}(-\bar{q}')$ the expression for $u^{(1)(-)}(-\bar{q}')$ given by (6) and for $u^{(r)(+)}(\bar{q})$ the expression (9). Thus the outgoing positron is assumed to be polarized along the $-z$ axis and the outgoing electron along the $+z'$ axis.

We obtain

$$\langle q, q' \,|\, M_4 \,|\, k \rangle = -\tfrac{1}{2}\left[(A_x + i\,A_y)\sin\frac{\alpha}{2} + (C + i\,A_z)\cos\frac{\alpha}{2}\right]\sqrt{\frac{(m+E)\,(m+E')}{4\,E\,E'}}, \tag{37}$$

where

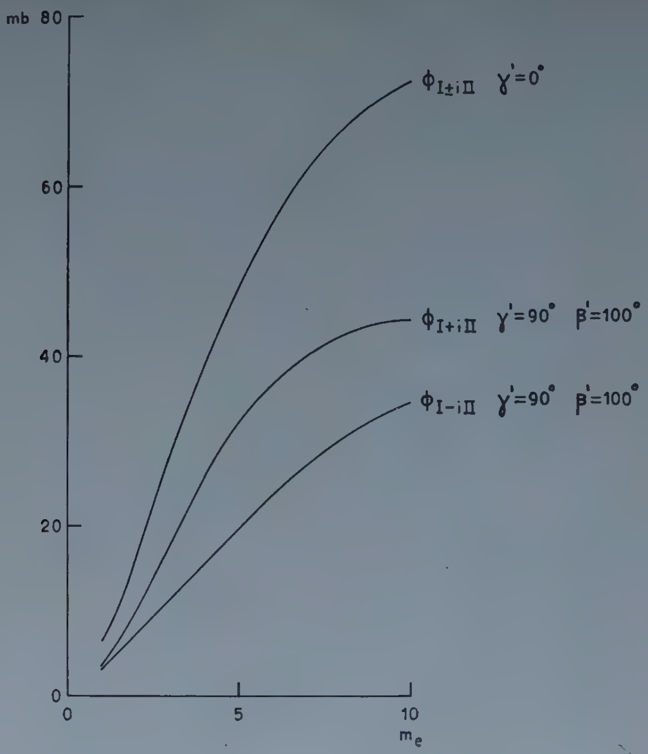

Fig. 7. $\omega = 0.5\,E$, $\gamma = 0°$.

$$\begin{aligned}\bar{A} = -\frac{m}{q\,k}\Bigg\{&-\bar{e}\,\omega - \frac{\bar{e}\,(\bar{q}'\cdot\bar{k}) - \bar{k}\,(\bar{e}\cdot\bar{q}') - 2\,\bar{q}'\,(\bar{e}\cdot\bar{q})}{m+E'} + \\ &+ \frac{\bar{e}\,(\bar{q}\cdot\bar{k}) - \bar{k}\,(\bar{e}\cdot\bar{q}) + 2\,\bar{q}\,(\bar{e}\cdot\bar{q})}{m+E} + \frac{\omega\,[\bar{e}\,(\bar{q}\cdot\bar{q}') - \bar{q}\,(\bar{e}\cdot\bar{q}') - \bar{q}'\,(\bar{e}\cdot\bar{q})]}{(m+E)\,(m+E')}\Bigg\} + \\ &+ \frac{m}{q'\,k}\Bigg\{-\bar{e}\,\omega + \frac{\bar{e}\,(\bar{q}'\cdot\bar{k}) - \bar{k}\,(\bar{e}\cdot\bar{q}') + 2\,\bar{q}'\,(\bar{e}\cdot\bar{q}')}{m+E'} - \\ &- \frac{\bar{e}\,(\bar{q}\cdot\bar{k}) - \bar{k}\,(\bar{e}\cdot\bar{q}) - 2\,\bar{q}\,(\bar{e}\cdot\bar{q}')}{m+E} + \frac{\omega\,[\bar{e}\,(\bar{q}\cdot\bar{q}') - \bar{q}\,(\bar{e}\cdot\bar{q}') - \bar{q}'\,(\bar{e}\cdot\bar{q}])}{(m+E)\,(m+E')}\Bigg\},\end{aligned} \tag{38}$$

$$C = \left[\frac{m}{q\,k} + \frac{m}{q'\,k}\right] \cdot \left[\frac{\bar{e}\cdot[\bar{q}\times\bar{k}]}{m+E} + \frac{\bar{e}\cdot[\bar{q}'\times\bar{k}]}{m+E'}\right] - \left[\frac{m}{q\,k} - \frac{m}{q'\,k}\right]\frac{\omega\,\bar{e}\cdot[\bar{q}'\times\bar{q}]}{(m+E)\,(m+E')}\,. \tag{39}$$

As previously we put $|\langle q, q'\,|\,M_4\,|\,k\rangle|^2 = \dfrac{(m+E)\,(m+E')}{16\,E\,E'}\cdot R.$ (40)

In this case the intensity of incoming particles is given by

$$I = \frac{1}{V} \tag{41}$$

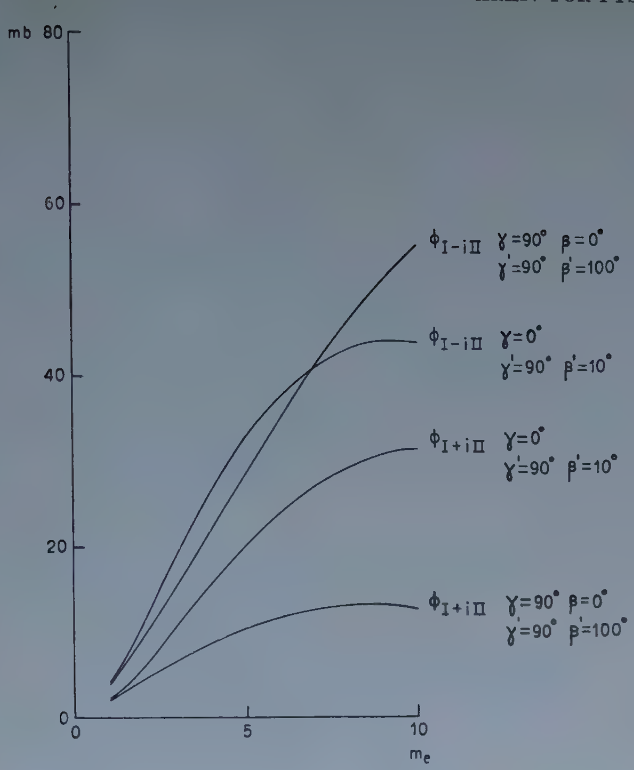

Fig. 8. $\omega = 0.5\,E$.

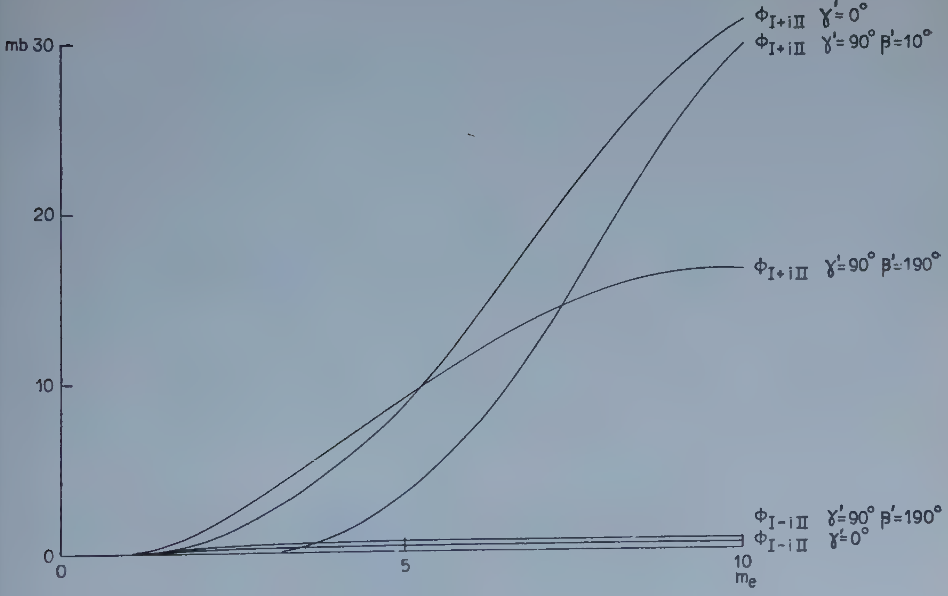

Fig. 9. $\omega = 0.9\,E$, $\gamma = 90°$, $\beta = 180°$; $\phi_{\text{I}-i\text{II}} < 0.005$, $\gamma' = 90°$, $\beta' = 10°$.

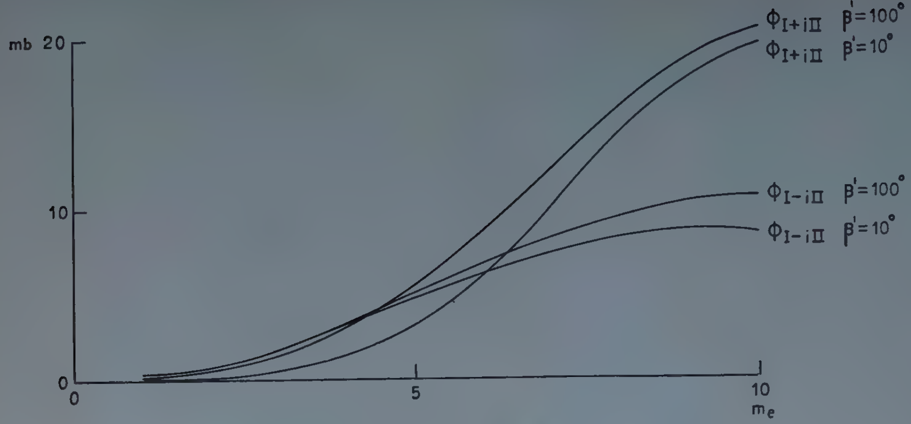

Fig. 10. $\omega = 0.9\,E$, $\gamma = 90°$, $\beta = 90°$, $\gamma' = 90°$.

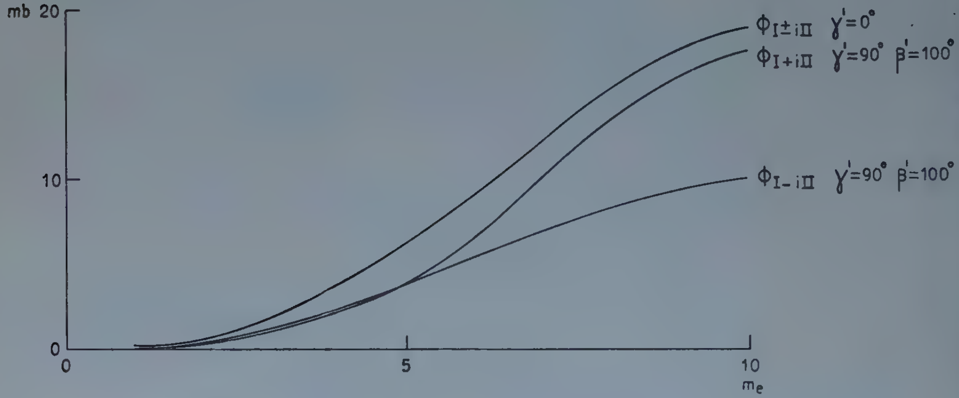

Fig. 11. $\omega = 0.9\,E$, $\gamma = 0°$.

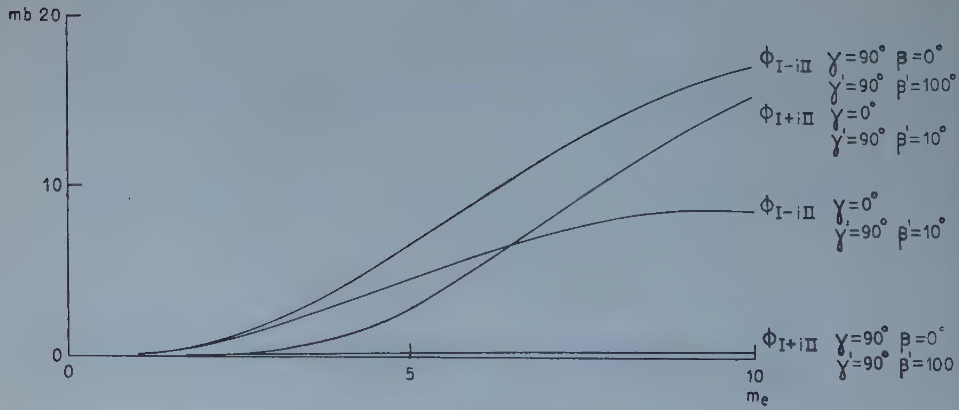

Fig. 12. $\omega = 0.9\,E$.

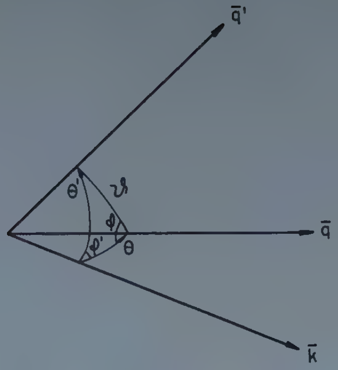

Fig. 13.

and the density of final states by

$$\varrho_F = \varrho(\bar{q})\,\varrho(\bar{q}')\,d E\,d E' = \frac{V^2}{(2\pi)^6}\,|\bar{q}|\cdot|\bar{q}'|\cdot E\cdot E'\,d\Omega_q\,d\Omega_{q'}\,d E\,d E', \tag{42}$$

and we thus obtain for the cross section

$$\begin{aligned} d\sigma &= \frac{Z^2 e^6}{(4\pi)^5 m^2}\cdot\frac{|\bar{q}|\cdot|\bar{q}'|\cdot(m+E)\,(m+E')}{\omega\,[(\bar{q}+\bar{q}'-\bar{k})^2]^2}\cdot R\cdot d E'\,d\Omega_q\,d\Omega_{q'} \\ &= Z^2\,\phi\cdot\frac{d E'}{E'-m}\,d\Omega_q\,d\Omega_{q'}. \end{aligned} \tag{43}$$

Here we have integrated over the energies of the electron, which means that according to the δ-function in (36) we have to make the substitution $E = \omega - E'$. The function ϕ depends on the energies of the photon and positron, of the polarization directions and of the angle θ between $\bar{k}$ and $\bar{q}$, θ' between $\bar{k}$ and $\bar{q}'$ and φ' between the $\bar{q}$, $\bar{k}$-plane and the $\bar{q}'$, $\bar{k}$-plane (Fig. 13).

On account of the similarity between (37) and (20) the function R in (43) is given by (30), (31) and (32) corresponding to linear or circular polarization of the photon. In this case, however, the $+$ sign in (31) refers to right circularly polarized photons. We may also write our formulae as functions of the same angles as before. The connections between θ', φ' and ϑ, φ are given by the well-known formulae

$$\left.\begin{aligned} \cos\vartheta &= \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos\varphi', \\ \sin\vartheta\sin\varphi &= \sin\theta'\sin\varphi'. \end{aligned}\right\} \tag{44}$$

We further write

$$\left.\begin{aligned} a_8 &= 2\,\frac{m\,|\bar{q}|\cdot|\bar{q}'|}{q'\,k\,(m+E)}, & a_9 &= 2\,\frac{m\,|\bar{q}|^2}{q\,k\,(m+E)}, \\ a_{10} &= 2\,\frac{m\,|\bar{q}'|^2}{q'\,k\,(m+E')}, & a_{11} &= 2\,\frac{m\,|\bar{q}|\cdot|\bar{q}'|}{q\,k\,(m+E')}. \end{aligned}\right\} \tag{45}$$

The components of A (38) and C (39) are now given by

$$\left.\begin{aligned}
A_{\mathrm{I}x_0} &= \sin\varphi \sin\vartheta\,(a_4\cos\theta - a_5 - a_8 - a_{10}\cos\vartheta),\\
A_{\mathrm{I}y_0} &= \sin\varphi\,(-a_0 + a_3\cos\theta - a_4\cos\theta\cos\vartheta + a_5\cos\vartheta - a_{10}\sin^2\vartheta),\\
A_{\mathrm{I}z_0} &= \cos\varphi\,(a_0 - a_3\cos\theta - a_5\cos\vartheta) + a_4\cos\theta''',\\
C_{\mathrm{I}} &= a_3\sin\theta + a_5\sin\vartheta\cos\varphi - a_4\cos\theta''
\end{aligned}\right\}\quad(46)$$

and

$$\left.\begin{aligned}
A_{\mathrm{II}x_0} &= (-a_0 + a_9 + a_{11}\cos\vartheta)\sin\theta - a_4\sin\vartheta\cos\varphi + (a_5 + a_8 + a_{10}\cos\vartheta)\cos\theta'',\\
A_{\mathrm{II}y_0} &= \cos\varphi\,(a_0\cos\theta - a_3 + a_4\cos\vartheta) - a_5\cos\theta''' + \sin\vartheta\,(a_{11}\sin\theta + a_{10}\cos\theta''),\\
A_{\mathrm{II}z_0} &= \sin\varphi\,(a_0\cos\theta - a_3 + a_4\cos\vartheta - a_5\cos\vartheta\cos\theta),\\
C_{\mathrm{II}} &= \sin\varphi\sin\vartheta\,(-a_4 + a_5\cos\theta).
\end{aligned}\right\}\quad(47)$$

We have here calculated $\phi_{\mathrm{I}\pm i\mathrm{II}}$ as function of φ' for the following values of the parameters.

$$\begin{aligned}
\omega - 2m &= 2,\ 3,\ 5,\ \text{and } 10\ m_e,\\
E' &= m + 0.5(\omega - 2m)\ \text{and}\ m + 0.9(\omega - 2m),\\
\theta &= 5^\circ,\ \theta' = 6^\circ.
\end{aligned}$$

γ	β	γ'	β'
0°		0°	
0°		90°	180° + ϑ
0°		90°	270° + ϑ
90°	180°	90°	270° + ϑ
90°	270°	90°	180° + ϑ

γ	β	γ'	β'
90°	270°	90°	270° + ϑ
90°	0°	90°	180° + ϑ
90°	0°	90°	ϑ
90°	0°	0°	

Figs. 14–15 show the curves obtained for backward polarization of the positron and forward polarization of the electron ($\gamma = 90^\circ$, $\beta = 0^\circ$, $\gamma' = 90^\circ$, $\beta' = \vartheta$) for the positron energy $E' = m + 0.5(\omega - 2m)$. The maximum value of the cross sections as function of the photon energy is shown in Figs. 16–23.

Finally, we want to say a few words about the determination of the cross sections for arbitrary polarization directions of the electrons. For simplicity we consider the case of linearly polarized photons. As mentioned earlier the polarization dependent part of the cross section is then given by (30). This formula contains besides the polarization directions ten quantities, viz. C_i^2, $A_{ix_0}^2$, $A_{iy_0}^2$, $A_{iz_0}^2$, $C_iA_{ix_0}$, $C_iA_{iy_0}$, $C_iA_{iz_0}$, $A_{ix_0}A_{iy_0}$, $A_{ix_0}A_{iz_0}$, $A_{iy_0}A_{iz_0}$. It is easily seen that if seven of these are known, we also know the remaining terms. Measuring or calculating the cross sections for nine combinations of the polarization directions of the electrons we obtain nine equations which are necessary and sufficient to determine the quantities above as functions of the known cross sections.

The numerical calculations have been made on the electronic computer of Lund University. A program for computation of the cross sections for all energies and scattering angles is available at the institute.

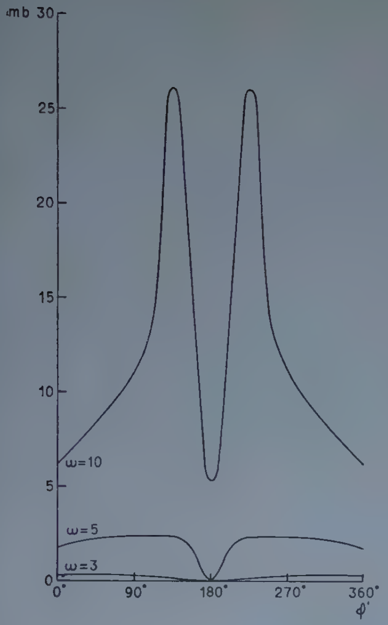

Fig. 14. $\phi_{\mathrm{I}-i\mathrm{II}}$, $E' = m + 0.5\,(\omega - 2m)$,
$\gamma = 90°$, $\beta = 0°$, $\gamma' = 90°$, $\beta' = \vartheta$.

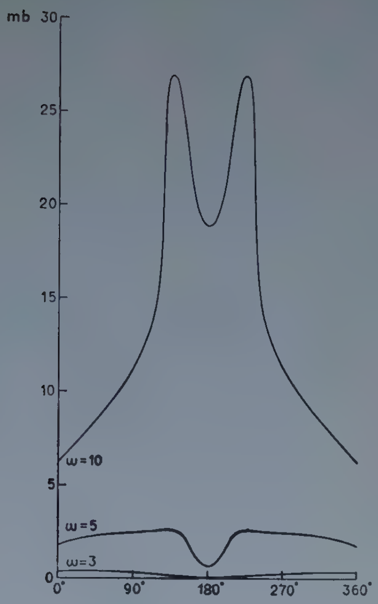

Fig. 15. $\phi_{\mathrm{I}+i\mathrm{II}}$, $E' = m + 0.5\,(\omega - 2m)$,
$\gamma = 90°$, $\beta = 0°$, $\gamma' = 90°$, $\beta' = \vartheta$.

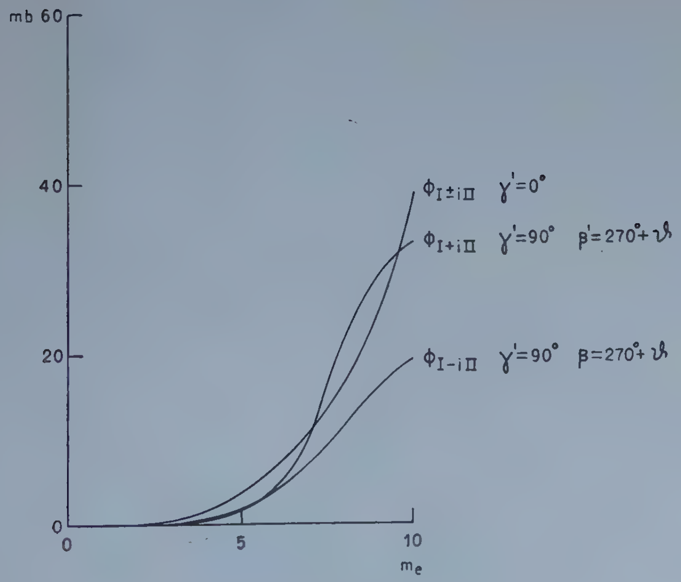

Fig. 16. $E' = m + 0.5\,(\omega - 2\mathrm{m})$, $\gamma = 0°$.

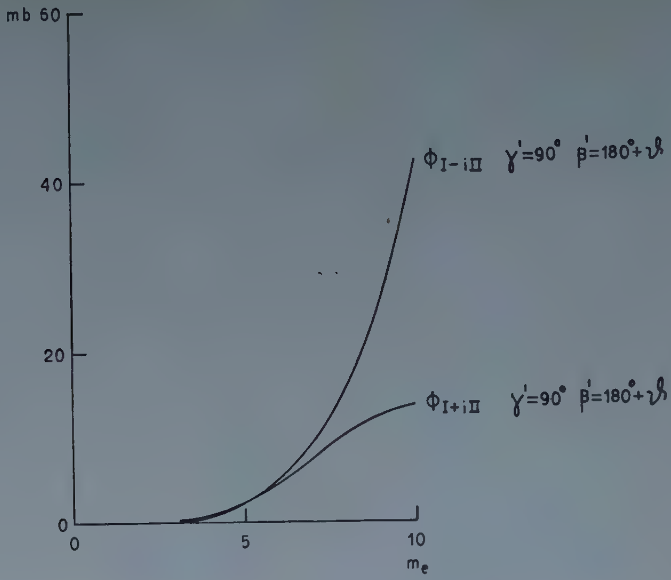

Fig. 17. $E' = m + 0.5(\omega - 2m)$, $\gamma = 0°$.

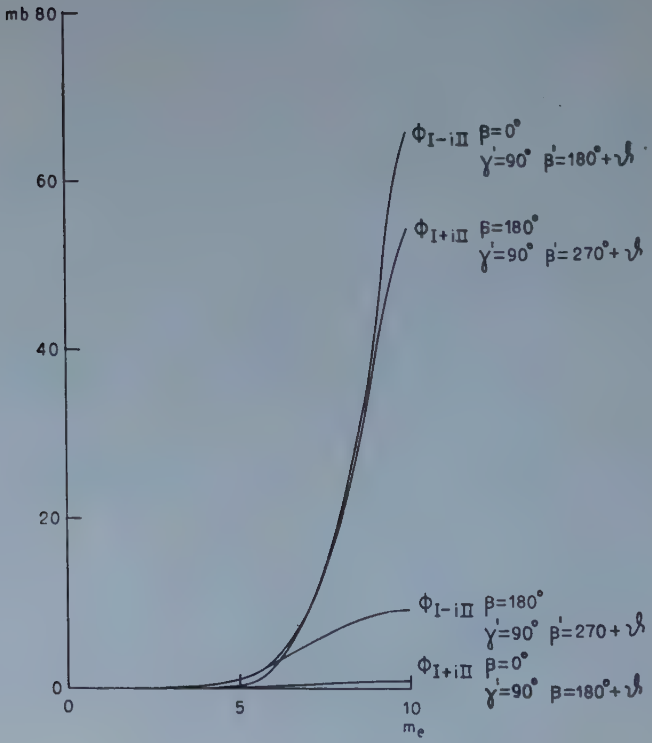

Fig. 18. $E' = m + 0.5(\omega - 2m)$, $\gamma = 90°$.

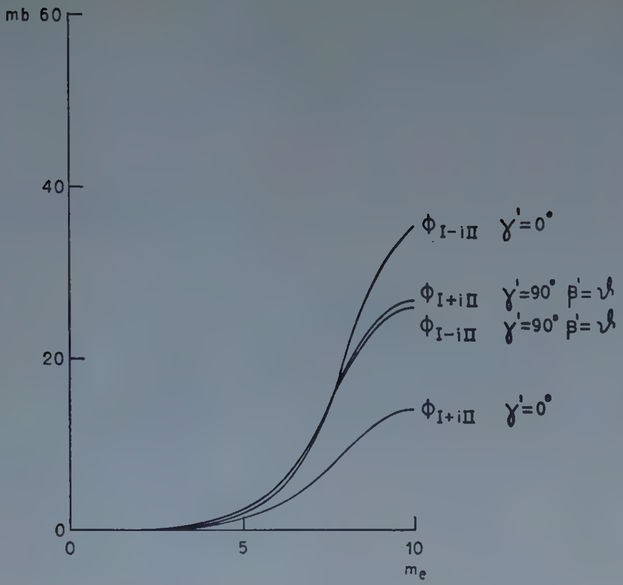

Fig. 19. $E' = m + 0.5(\omega - 2m)$, $\gamma = 90°$, $\beta = 0°$.

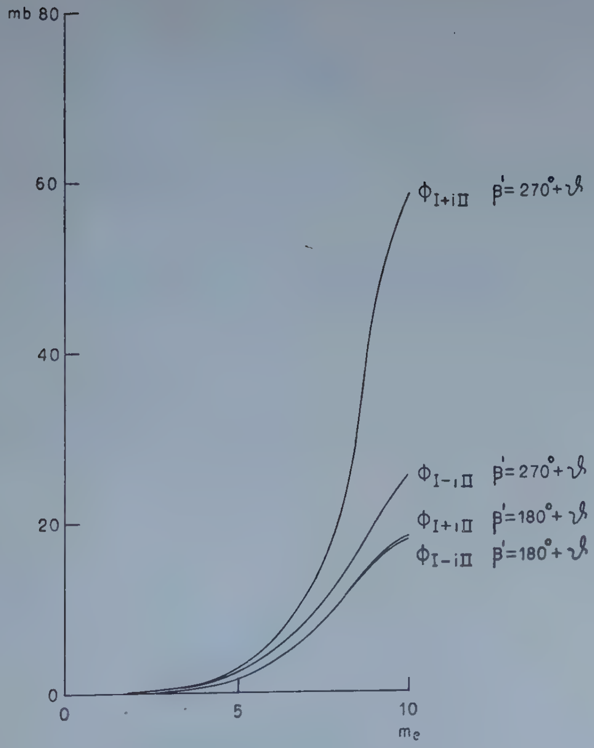

Fig. 20. $E' = m + 0.5(\omega - 2m)$, $\gamma = 90°$, $\beta = 270°$, $\gamma' = 90°$.

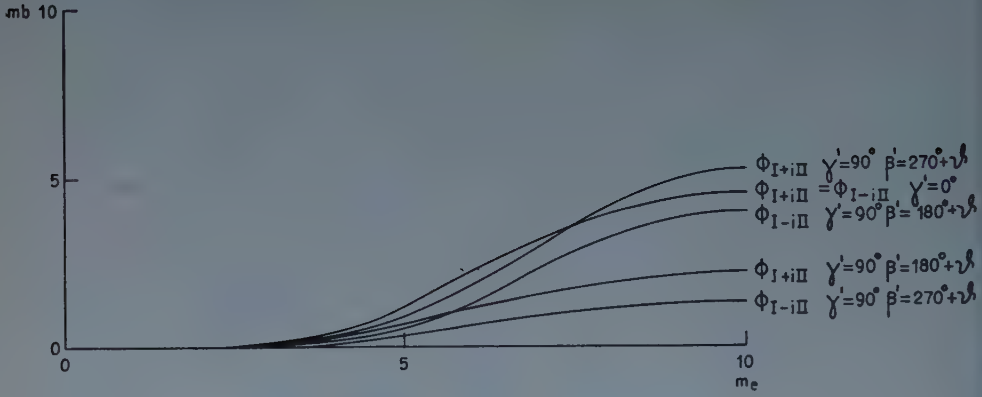

Fig. 21. $E' = m + 0.9(\omega - 2m)$, $\gamma = 0°$.

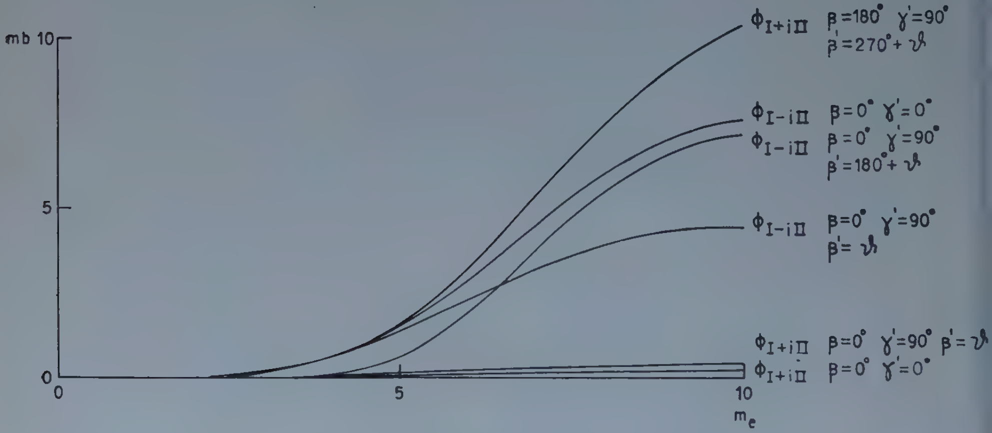

Fig. 22. $E' = m + 0.9(\omega - 2m)$, $\gamma = 90°$;
$\phi_{I+iII} < 0.06$, $\beta = 0°$, $\gamma' = 90°$, $\beta' = 180° + \vartheta$;
$\phi_{I-iII} < 0.09$, $\beta = 180°$, $\gamma' = 90°$, $\beta' = 270° + \vartheta$.

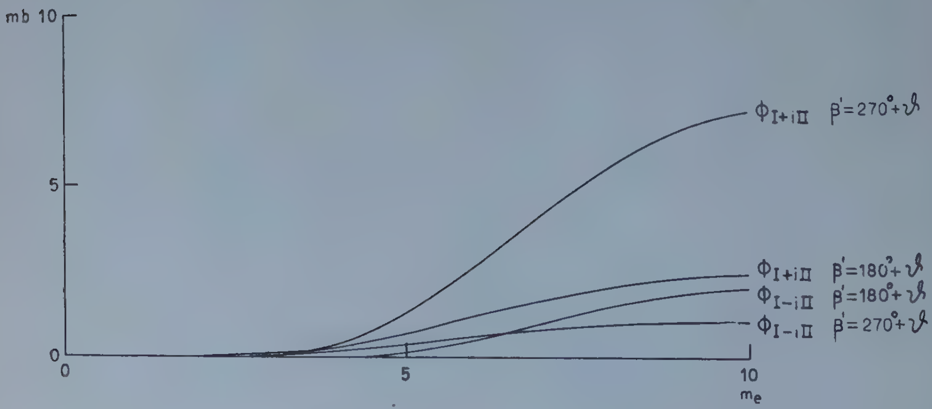

Fig. 23. $E' = m + 0.9(\omega - 2m)$, $\gamma = 90°$, $\beta = 270°$, $\gamma' = 90°$.

ACKNOWLEDGEMENTS

I want to express my gratitude to Professor T. Gustafson for the kind interest he has shown in this work. I am further indebted to Dr. G. Källén for suggesting this investigation and for many valuable discussions. My thanks are also due to Dr. C.-E. Fröberg and Mr. K. Jönsson who made it possible for me to perform the numerical calculations on SMIL, the electronic computer of Lund University.

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Tryckt den 23 oktober 1957

Uppsala 1957. Almqvist & Wiksells Boktryckeri AB

ERRATA

E. RUNE LINDGREN, **The transition process and other phenomena in viscous flow (this volume, pp. 1—169).**

Page 36, line 10: *For* the circulation was started *read* the circulation pump was started

88, Fig. 4.10, diagram 3, 12.31 mm tube. *The black dot and the open circle given for the Reynolds number 2270 ought to change place internally as also the black dot and the open circle given for the Reynolds number 2350. (Compare Table 4.2 on pp. 86–87.)*

88, line 5 from bottom: *For* reported in Part 1. *read* reported in Part 2.

102, line 8: *For* Reynolds number where each *read* Reynolds number above which each

104, Fig. 5.1: *The ordinate axes are marked erroneously. (Compare Fig. 4.4 on p. 79.)*

105, line 9 from bottom: *For* maximum presure difference *read* maximum pressure difference

123, line 7: *For* pressure drops *read* pressure drop

143, line 9: *For* There are quite of few *read* There are quite a few

147, line 14: *For* R. R. Hughes & A. K. Oppenheimer 1956 *read* R. R. Hughes & A. K. Oppenheim 1956

158, line 22: *For* pressure centre anon d this *read* pressure centre and on this

It should also be noted that, owing to a linguistic misunderstanding, the term "vorticity" has repeatedly been used in the meaning of "vortices".